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Magnetic Fields in the Early Universe

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We discuss magnetic fields in the early universe—their origin, their possible detection, and limits and values today and at early times.

1. INTRODUCTION

We discuss questions related to magnetic fields and possible ways to observe them at earlier times. We start with nucleosynthesis, since ideas about the chiral transition time or even the electroweak transition are not fully understood. We then discuss magnetic fields at recombination time and the possibility to measure these fields, if they existed, via the coming cosmic microwave background radiation (CMBR) experiments.

We finally discuss what is known and perhaps known starting at $z = 6$ until present times.

2. MAGNETIC FIELDS AT BBN TIME

In this section we discuss limits on magnetic fields that could have been present at nucleosynthesis time. We consider several effects that could be relevant modifing the relic abundances of light elements. They include changes in reaction rates, mass shifts due to strong and electromagnetic interactions, and variation of the expansion rate of the universe due to both the magnetic field energy density and the increasing of the electron density in overcritical magnetic fields. We find that the latter is the main effect. It was not taken into account in previous calculations. The allowed field intensity at the end of nucleosynthesis ($T = 1 \times 10^9$ K) is $B \leq 3 \times 10^{10}$ G.

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Among the many uncertainties in the early universe environment, the possibility that large, constant magnetic fields existed over macroscopic scales is a fascinating possibility. Since the early universe is believed to be a perfect conductor, magnetic lines get thinned out by the ratio

$$\frac{B_1}{B_2} = \left(\frac{R_2}{R_1}\right)^2 \tag{1}$$

The presence of primordial fields over galactic scales, when extrapolated back, can give, under different assumptions, very large fields indeed.

A typical present-day galactic field $B = 10^{-6}$ G can grow, scaled as dictated by Eq. (1) alone, to be as large as 10^{14} G. These extrapolations are very doubtful since dynamo effects may have enlarged significantly present fields. Nevertheless, knowing the allowed fields at a given epoch and limiting its value at another one can give important dynamical restrictions.

The size of the patch at the nucleosynthesis time might allow for large fields, though we have no real reliable model for the field evolution.

In this paper we address the allowed magnetic fields at nucleosynthesis time, without discussing their origin. We give a detailed analysis of the influence of the fields on the main quantities that can act to modify the relic elements abundance ratios: reaction rates, masses of the participants, electron energy densities, and magnetic field energy density. We then obtain, given the present errors in these relative abundances, the upper limits the fields can take in regions large compared to the reaction scale, but possibly much smaller than the horizon at that time. In the next section we discuss briefly the impact of the magnetic field on these elements of the calculation. Then we describe the standard nucleosynthesis calculations in this light. In the final section we discuss the constraints and their origin and compare with existing calculations.

3. WEAK REACTION RATES IN THE PRESENCE OF MAGNETIC FIELDS

The main weak processes which act to determine the n/p ratio during the the primordial nucleosynthesis are

$$n + e^+ \rightleftharpoons p + \bar{\nu} \tag{a}$$

$$n + \nu \rightleftharpoons p + e^- \tag{b}$$

$$n \rightleftharpoons p + e^- + \bar{\nu} \tag{c}$$

The rate for two-body scattering reactions in a medium may be written in the form

$$\Gamma(12 \to 34) = \left(\prod_i \frac{\int d^3\mathbf{p}_i}{(2\pi)^3 2E_i} \right) (2\pi)^4 \delta^4\Big(\sum_i p_i\Big) |\mathcal{M}|^2 f_1 f_2 (1 - f_3)(1 - f_4) \tag{2}$$

where p_i is the four-momentum, E_i is the energy, and f_i is the number density of each particle species. All processes (a)–(c) have the same amplitude

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \cos\theta_C \, \bar{u}_p \gamma_\alpha (1 - \alpha\gamma_5) u_n \bar{u}_e \gamma_\alpha (1 - \gamma_5) u_\nu \tag{3}$$

where $\alpha = g_A/g_V \simeq = -1.262$. Without any external magnetic field the total rate of the processes that convert neutrons to protons is

$$\Gamma_{n\to p}(B = 0) = \frac{1}{\tau} \int_1^\infty d\epsilon \, \frac{\epsilon\sqrt{\epsilon^2 - 1}}{1 + e^{m_e\epsilon/T + \phi_e}} \times \left[\frac{(q + \epsilon)^2 \, e^{(\epsilon+q)m_e/T_\nu}}{1 + e^{(\epsilon+q)m_e/T_\nu}} + \frac{(\epsilon - q)^2 \, e^{\epsilon m_e/T + \phi_e}}{1 + e^{(\epsilon-g)m_e/T_\nu}} \right] \tag{4}$$

where $1/\tau \equiv G^2(1 + 3\alpha^2)m_e^5/2\pi^3$ and q, ϵ, and ϕ_e are, respectively, the neutron–proton mass difference, the electron energy, and the electron chemical potential, all expressed in units of m_e. We assume that the chemical potential of the neutrinos is vanishing.

The total rate for the $p \to n$ processes can be obtained by changing the sign of q in Eq. (4).

An external magnetic field leads us to modify Eq. (4) due to the following effects.

(i) The dispersion relation of charged particles propagating through a magnetic field is modified with respect to the free-field case. In fact, their 4-momentum in this case is $p = p(B = 0) + qA$, where q is the charge of the particle and the vector potential $\mathbf{A}(\mathbf{r})$ is related to the field by $\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{r} \times \mathbf{B}$. Assuming $\mathbf{B}$ along the $\mathbf{z}$ axis, the expressions for energies of electrons, protons and neutrons are, respectively,

$$E_e = [p_{e,z}^2 + eB(2n + 1 + s) + m_e^2]^{1/2} + \kappa \tag{5}$$

$$E_p = [p_{p,z}^2 + eB(2n + 1 - s) + m_p^2]^{1/2} - \frac{e}{2m_p}\left(\frac{g_p}{2} - 1\right) B \tag{6}$$

$$E_n = [\mathbf{p}_n^2 + m_n^2]^{1/2} + \frac{e}{2m_n}\frac{g_n}{2} B \tag{7}$$

In the above, n denotes the Landau level, $s = \pm 1$ indicates whether the spin is along or opposed to the field direction, and $g_p = 5.58$ and $g_n = -3.82$

are the Landé g-factors. The QED correction to the electron energy κ was first computed by Schwinger [1]. For magnetic fields larger than $\sim 10^{13}$ G this correction is

$$\kappa = \frac{\alpha}{2\pi} \ln\left(\frac{2eB}{m_e^2}\right)^2 \tag{8}$$

For smaller field intensity κ has negligible effects on our calculations and we disregard it. The effects of the field on the QCD ground state have been parametrized via a field-dependent nucleon mass [2], as we are going to discuss below.

Neither the neutron nor the neutrino has quantized levels, though the neutron has an electromagnetic interaction energy. The neutrino is totally inert *vis à vis* electromagnetism.

(ii) The number of available states for a particle obeying Eq. (4) becomes, for every value of n and s [3],

$$\frac{VeB}{(2\pi)^2}\, dp_z \tag{9}$$

This changes the phase space of the processes in which we are interested.

(iii) Since the occupation number and the energy of states with opposite spin projections is not the same in a magnetic field, the spin sum of the square amplitude needs to be weighted by the appropriate spin-dependent Fermi distributions.

Nucleosynthesis take place in a range of temperatures $0.1 < T < 10$ MeV, hence nucleons are nonrelativistic. Therefore nucleon distribution functions are given by

$$f_N(s = \pm 1) = (1 + e^{\mp \mu_N B/T})^{-1} \tag{10}$$

where

$$\mu_p = \frac{e}{2m_p}\frac{g_p}{2}, \qquad \mu_n = \frac{e}{2m_n}\frac{g_n}{2} \tag{11}$$

Since during the nucleosynthesis $m_N \gg T$, and momenta are also small compared to nucleon mass, f_N can be safely approximated by 1/2. This is not the case for electrons. In this case we have

$$f_e(s) = (1 + e^{E_e(s)/T})^{-1} \tag{12}$$

where the relativistic expression for the electron energy, Eq. (5), is used.

As a consequence, the integral for the leptonic momentum space in neutron β decay is modified to

$$\frac{1}{2\pi}\sum_{n=0}^{N_c}\int_{-\infty}^{\infty}\frac{d^3\mathbf{p}_\nu}{2E_\nu}\int_{-p_{e,z}(n)}^{p_{e,z}(n)}\frac{dP_{e,z}}{2E_e}\,eB|\mathcal{M}|^2(1-f_\nu)(1-f_e) \tag{13}$$

where N_c is largest integer n such that $p_{e,z}(n)^2 = Q^2 - m_e^2 - 2neB$ is positive and $Q^2 \equiv m_n^2 - m_p^2$.

(iv) The nucleon masses are affected by very strong magnetic fields. The change in effective phase space is [2]

$$\Delta = 0.12\mu_N B - M_n + M_p + f(B) \tag{14}$$

The function $f(B)$ gives the rate of mass change due to color forces being affected by the field. For nucleons [2] the main change is the chiral condensate growth, which because of the different quark content of protons and neutrons makes the proton mass grow faster [2]. Though the sign is certain, vacuum pairs of zero helicity get more bound in the presence of a B field; the size of the effect is model dependent. We have calculated its influence using the weakest and strongest reasonable field dependence and we find that the effect is always small for fields below 10^{18} G.

Having established that hadronic mass changes will not affect nucleosynthesis, we drop these effects from the equations altogether.

Taking into account the remaining effects, we computed the total rate for the weak processes converting neutrons to protons in an external magnetic field. The result is

$$\Gamma_{n\to p}(B) = \frac{\gamma}{\tau}\sum_{n=0}^{\infty}(2-\delta_{n0})$$
$$\times\int_{\sqrt{1+2(n+1)\gamma+\kappa}}^{\infty} d\epsilon\,\frac{(\epsilon-\kappa)}{\sqrt{(\epsilon=\kappa)^2-1-2(n+1)\gamma}}$$
$$\times\frac{1}{1+e^{m_e\epsilon/T+\phi_e}}\left[\frac{(\epsilon+g)s2e^{m_e(\epsilon+g)/T_\nu}}{1+e^{m_e(\epsilon+g/T_\nu+\phi_e}}+\frac{(\epsilon-q)^2e^{m_e\epsilon/T+\phi_e}}{1+e^{m_e(\epsilon-g)/T_\nu}}\right] \tag{15}$$

where $\gamma \equiv B/B_c$ and $B_c = m_e^2/e = 4.4\times10^{13}$ G is usually defined to be the critical magnetic field.

Equation (15) coincides with the result of Matese and O'Connell [4] and Cheng *et al.* [16] in the limit in which the QED correction κ goes to zero. Although the quantitative effects of this term on the nucleosynthesis predictions are subdominant, we stress that disregarding it when the field is overcritical leads to an unstable electron ground state, thus to unphysical results.

The main effect of the magnetic field is due to the modification of the electron phase space. Equation (15) is correct in the weak-field limit, when $\gamma \ll 1$ and $\Gamma_{n\to p}(B)$ reduces to Eq. (4) in the $B = 0$ limit.

For large values of γ and fixed temperature, the total rate grows like γ. Increasing the temperature, the relevant contribution to the integrals in Eq. (15) comes from the high-energy part of the electron spectrum. Since the limit $\epsilon \to \infty$ is equivalent to the limit $\gamma \to 0$ in Eq. (15), this explains why the ratio $\Gamma_{n\to p}(B)/\Gamma_{n\to p}(0)$ goes to one when $T \gg m_e$. Although the global rate of the inverse process $\Gamma_{p\to n}$ also increases with B, it remains suppressed by a factor $\exp(-Q(B)/T)$ with respect to $\Gamma_{n\to p}$. Thus the effect of a strong magnetic field would be to *reduce* the final number of neutrons in the universe, i.e., the relic ^{4}He abundance, if only the correction to the weak rates is taken into account.

4. THE EFFECTS OF *B* ON THE EXPANSION RATE

Owing to exponential dependence of the (n/p) equilibrium ratio on the temperature, the relic relative abundances of light elements depends crucially on the freezeout temperature T_F of the weak processes that keep protons and neutrons in chemical equilibrium.[2] This temperature is essentially determined by the condition

$$\Gamma_{n\rightleftharpoons p}(T_F) = H(T_F) \tag{16}$$

where H is the expansion rate of the universe.

It is evident that besides the rate of the weak processes we need to pay attention to the effects of the magnetic field on H. If no cosmological constant is present, the expansion rate is determined by the Einstein equation

$$H^2(T) = \frac{8\pi G_N}{3}\rho(T) \tag{17}$$

where $\rho(T)$ is the total energy density of the universe. In the case that no magnetic field is present, $\rho(T)$ is given by the sum of the energy density of all the particle species in thermal equilibrium with the primordial plasma

$$\rho(T) = \rho_\gamma(T) + \rho_e(T) + \rho_\nu(T) + \rho_b(T) \tag{18}$$

where the subscripts γ, e, ν, b stand, respectively, for photons, electrons, the three species of neutrinos, and baryons, including their respective antiparticles. In our case, since the magnetic field has energy density $\rho_B(T) = B(T)^2/8\pi$, this term also needs to be added to Eq. (18). Since we have magnetic flux conservation in the plasma

$$B \propto R^{-2} \propto T^2$$

the energy density of the magnetic field has the same temperature dependence as the energy density of the radiation.

[2] See ref. 17 for a review of of most aspects of primordial nucleosynthesis big bang cosmology.

This new contribution to $\rho(T)$ will dominate over the other terms in Eq. (18) if

$$B(T = 10^{11}\ \mathrm{K}) \gtrsim 10^{16}\ \mathrm{G} \tag{19}$$

We assumed the pressure associated with the random magnetic field to be zero on average. Although a novanishing mean pressure is also possible for random magnetic fields [18], our final conclusions are not affected also taking this pressure into account.

The presence of a nonvanishing ρ_B is not the only effect that modifies the expansion rate of the universe. The energy density of charged particles in the primordial plasma is also affected. In the previous section we have showed how the electron dispersion relation and the electron phase space are modified by the magnetic field. Using Eqs. (5) and (9), we get the electron energy density as function of γ,

$$\begin{aligned}\rho_e(T) &= \frac{eB}{2\pi^2}\sum_{n=0,s}^{\infty}\int dp_z\, E_e(s) f_e \\ &= \frac{\gamma}{2\pi^2}\sum_{n=0}^{\infty}(2-\delta_{n0})\int_{\sqrt{1+2(n+1)\gamma+\kappa}}^{\infty} d\epsilon \\ &\quad\times \epsilon\,\frac{(\epsilon-\kappa)}{\sqrt{(\epsilon-\kappa)^2-1-2(n+1)\gamma}}\,\frac{1}{1+e^{m_e\epsilon/T+\phi_e}}\end{aligned}$$

One can see from Fig. 1 that the effect of an overcritical magnetic given by Eq. (19) is to increase the electron energy density roughly linearly with the field intensity. An equation like Eq. (20) is valid for positrons once the sign of the chemical potential ϕ_e is changed. In analogy with what happen for the reaction rates, this effect becomes less important at high temperatures if B is left fixed. However, it is a very relevant effect when $T \lesssim 1$ MeV. Although the field intensity, hence the correction to ρ_e, decreases like T^2, we are going to show that this is the main effect on the primordial nucleosynthesis predictions.

5. CONCLUSIONS ABOUT BIG BANG NUCLEOSYNTHESIS

In the previous sections we have shown that the existence of large magnetic fields during the primordial nucleosynthesis affects the final relative abundances of the light elements via two main effects: (a) the increasing of the weak reaction rates and (b) the increasing of the expansion rate of the universe. These are competing effects. In fact, whereas the former tends to reduce the (n/p) freezeout temperature, hence the final abundance of ^{4}He, the latter acts in the opposite direction.

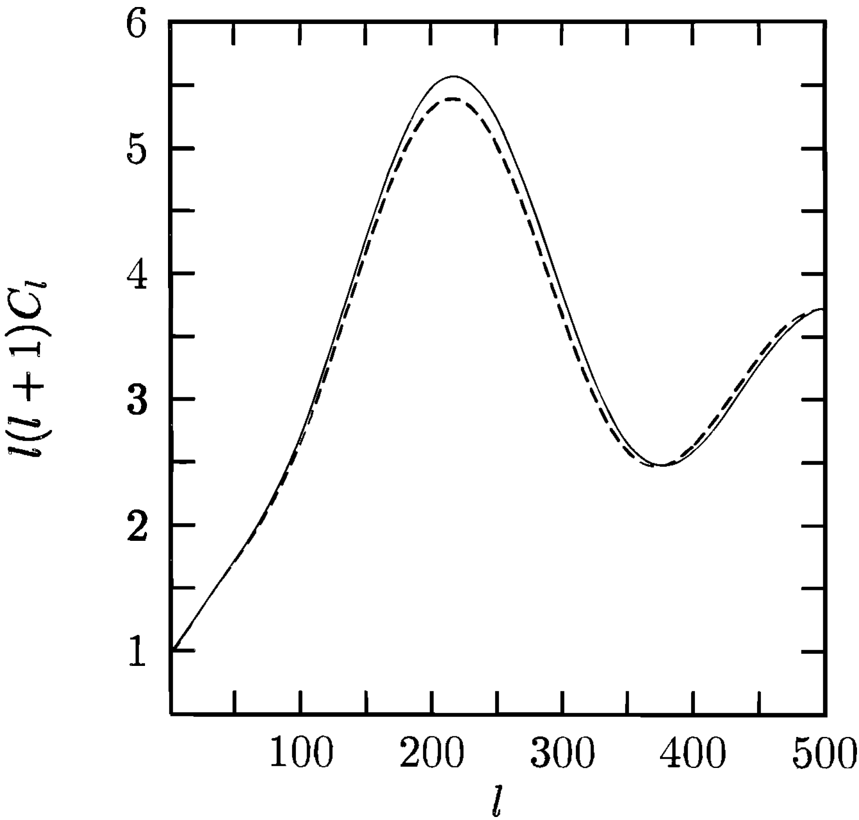

Fig. 1. The effect of a cosmic magnetic field on the multipole moments. The solid line shows the prediction of a standard CDM cosmology ($\Omega = 1$, $h = 0.5$, $\Omega_B = 0.05$) with an $n = 1$ primordial spectrum of adiabatic fluctuations. The dashed line shows the effect of adding a magnetic field equivalent to 2×10^{-7} G today.

We modified the standard nucleosynthesis code [19][3] to take into account all the relevant effects, as well as other effects that eventually we neglected as irrelevant. Since our aim is to get an upper limit to the magnetic field intensity, we adjusted the value of the baryon photon ratio η in order to get the minimal ^{4}He relic abundance prediction compatible with observations [21] in the free-field case. Table I presents our predictions for some light element relic abundances. As is evident, the limits on magnetic fields are totally controlled by the ^{4}He abundance.

Other elements reach forbidden values only at very high fields, in which case the effect of mass changes due to color forces will also be important.

[3] Cancel e is a modernized and optimized version of the code written by Wagoner [20].

Table I. Predictions of Light Element Abundances at the End of the Primordial Nucleosynthesis for Several Values of the Magnetic Field Intensity at the Temperature $T = 10^{11}$ K

$B(T = 10^{11}$ K)	^{4}He	(D + ^{3}He)/H	^{7}Li/H
0	0.236	1.14×10^{-4}	1.11×10^{-10}
1×10^{12}	0.236	1.14×10^{-4}	1.11×10^{-10}
5×10^{12}	0.236	1.14×10^{-4}	1.11×10^{-10}
1×10^{13}	0.237	1.13×10^{-4}	1.11×10^{-10}
5×10^{13}	0.240	1.08×10^{-4}	1.14×10^{-10}
1×10^{14}	0.242	1.05×10^{-4}	1.15×10^{-10}
5×10^{14}	0.247	9.99×10^{-5}	1.20×10^{-10}
1×10^{15}	0.250	9.71×10^{-5}	1.23×10^{-10}
5×10^{15}	0.257	9.15×10^{-5}	1.32×10^{-10}
1×10^{16}	0.348	8.92×10^{-5}	1.35×10^{-10}

Fields larger than 10^{10} G at the end of nucleosynthesis are therefore not allowed. Our calculations are compatible with estimates of expected fields at that time.

The increase of the ^{4}He relic abundance with the field intensity reveals that the effect of B on the expansion rate is the most relevant. Regarding this point we agree with the qualitative conclusion of Matese and O'Connell [4] and disagree with the opposite conclusion of Cheng *et al.* [22]. Mainly, we do not understand how they can reconcile the claim that the dominant effects of the magnetic field are those arising from modification of the reaction rates with the growing of the relic ^{4}He that they get increasing B.

Furthermore, in both refs. 4 and 22 the effect of the magnetic field on the electron and positron energy density was not considered. Leaving only this effect on in our code, we checked that this is indeed the most relevant. We showed that this is indeed the main effect as long as the field intensity at the beginning of the nucleosynthesis is smaller than 10^{16} G.

Since the observational upper bound for the ^{4}He relic abundance is $Y_p \leq 0.245$, we conclude that the average intensity of a random magnetic field at the temperature of $T = 1 \times 10^{11}$ K (beginning of nucleosynthesis) must be less than 3×10^{14} G or, equivalently, $B(T = 10^9\ \mathrm{K}) < 3 \times 10^{10}$ G (end of nucleosynthesis). Vachaspati [6] predicts a magnetic field strength of $\sim 10^{11}$ G, on the smallest coherence region of the field ($N = 1$), at the end of nucleosynthesis. This extreme assumption ($N = 1$) is ruled out by our limits.

Assuming the field continues to rescale according to Eq. (1) (perhaps not a reasonable assumption), our results imply that the intergalactic field is less than $\sim 3 \times 10^{-7}$ G at present.

The other light element relic abundances are less affected than ^{4}He by the magnetic field and we do not use them to get constraints. However, it is interesting to observe the behavior of deuterium and ^{3}He abundances versus the initial magnetic field. Although they increase with the field at the beginning of nucleosynthesis, their relic abundances follow the opposite behavior. This can be understood since the rates of the processes converting deuterium and ^{3}He to ^{4}He are proportional to the initial abundances Y_D or $Y_{3\mathrm{He}}$ [17]. The greater are the rates, the smaller are the freezeout temperatures for these light elements. Thus we expect smaller D and ^{3}He relic abundances even if the relic ^{4}He increases. For details of these calculations see ref. 15 and references therein.

6. FIELDS AT RECOMBINATION TIMES AND FIELDS TODAY

In this section we study the effect of a magnetic field on the fluctuation spectrum of the cosmic microwave background. We find that upcoming measurements might give interesting bounds on large-scale magnetic fields in the early universe. If the effects are seen, it might be possible to establish the presence of different fields in different patches of the sky. Absence of any effect will provide by one order of magnitude a better limit for a primordial field, now given by nucleosynthesis. Even the stability of large fields is open to conjecture [5]. In the galaxy one measures a field of the order of 10^{-6} G, but its origin remains a mystery [6]. If it is primordial, it could have resulted from a compression of a cosmological field corresponding to around 10^{-9} G today. This is comparable to limits set for fields on the horizon scale using Faraday rotation on faraway galaxies and quasars. When traced back in time such a field becomes quite strong since $B \sim 1/a^2$, where a is the scale factor.

The presence of primordial fields is a hotly debated issue. For a long time the dynamo mechanism with small seed fields was favored, but the recent discovery of damped Lyα lines in QSOs indicates that primordial fields existed at early times. Moreover, there are problems with the dynamo mechanism. For a short discussion and further references see ref. 7.

The QSO measurements are consistent with having μG fields at $z^{abs} = 2$. It is not unreasonable to expect that such fields might have had measurable effects on physics in the early universe. One such possibility was studied in ref. 15, where it was found that nucleosynthesis bounded the field to 10^{11}–10^{12} G (lower limit for fields homogeneous on the horizon scale) at a time when $T = 10^9$ K. This corresponds to between 10^{-6} and 10^{-7} G today. Another way to set limits, this time at last scattering, is to study Faraday rotation directly in the CMB. In ref. 7 it is claimed that it should be possible to reach a field equivalent to 10^{-9} G today in this way. Existence of these fields may also have a large impact on structure formation [11].

In the remaining sections we will discuss the possibility of taking advantage of the many upcoming precision measurements of CMB anisotropies.

These measurements, involving satellites, ground interferometry, and balloons,[4] promise to provide us with accurate values of many cosmological parameters.

When primordial density fluctuations, perhaps generated by inflation, enter the horizon some time before last scattering, they initiate acoustic oscillations in the plasma. These oscillations distort the primordial spectrum of fluctuations and their effect can be studied today. Clearly the result will be very sensitive to the physics of the plasma and this is the reason for the present optimism.

As we will argue here, magnetic fields of reasonable magnitude will also affect the plasma leaving a possibly measurable imprint on the CMB. There are several exciting possibilities that may be detectable: (a) different types of waves (see below) depending on the properties of the primordial fluid creating different displacements of acoustic peaks and changing their magnitudes, and (b) anisotropies (at the level of 10^{-6}) that may be different in different areas of the sky, signaling the presence of magnetic field patches in the early universe.

7. SOME MAGNETOHYDRODYNAMICS

A rigorous analysis of the effects of the magnetohydrodynamic modes on the CMB requires the introduction of a multifluid theory and a general relativistic treatment. However, a brief description of the main features of the magnetohydrodynamics of a nonrelativistic one-component charged fluid is physically illuminating and will occupy this section.

We will consider a magnetic field homogenous on scales larger than the scale of plasma oscillations. We will therefore assume a background magnetic field $\mathbf{B}_0$ constant in space. The actual field is $\mathbf{B}_0 + \mathbf{B}_1$, where $\mathbf{B}_1$ is a small perturbation. We assume that the electric conductivity of the medium is infinite, thus the magnetic flux is constant in time. Then, due to the expansion of the universe, $B_0 \propto a^{-2}$. We neglect here any dissipative effect due, for example, to a finite viscosity and heat conductivity [11]. In other words we are assuming that $\lambda = 2\pi/k \gg l_{\text{diss}}$. This is justified for the large-scale fields that we are considering.

Within these assumptions the linearized equations of MHD in comoving coordinates are

$$\delta + \frac{\nabla \cdot \mathbf{v}_1}{a} = 0 \tag{21}$$

$$\dot{\mathbf{v}}_1 + \frac{\dot{a}}{a}\mathbf{v}_1 + \frac{c_S^2}{a}\nabla\delta + \frac{\nabla\phi_1}{a} + \frac{\hat{\mathbf{B}}_0 \times (\dot{\mathbf{v}}_1 \times \hat{\mathbf{B}}_0)}{4\pi a^4} + \frac{\hat{\mathbf{B}}_0 \times (\nabla \times \hat{\mathbf{B}}_1)}{4\pi\rho_0 a^5} = 0 \tag{22}$$

[4] See ref. 10 for references to planned and ongoing experiments.

$$\partial_t \hat{\mathbf{B}}_1 = \frac{\nabla \times (\mathbf{v}_1 \times \hat{\mathbf{B}}_0)}{a} \tag{23}$$

$$\nabla^2 \phi_1 = 4\pi G \rho_0 \left(\delta + \frac{\hat{\mathbf{B}}_0 \cdot \hat{\mathbf{B}}_1}{4\pi \rho_0 a^4} \right) \tag{24}$$

and

$$\nabla \cdot \hat{\mathbf{B}}_1 = 0 \tag{25}$$

where $\hat{\mathbf{B}} \equiv \mathbf{B}a^2$ and $\delta = \rho_1/\rho_0$, ϕ_1 and v_1 are small perturbations on the background density, gravitational potential, and velocity, respectively. c_S is the sound velocity. Neglecting its direct gravitational influence, the magnetic field couples to fluid dynamics only through the last two terms in Eq. (22). The first of these terms is due to the displacement current contribution to $\nabla \times \mathbf{B}$, whereas the latter account for the magnetic force of the current density. The displacement current term can be neglected provided that $v_A = B_0/\sqrt{4\pi\rho} \ll c_S$, where v_A is the Alfvén velocity.

Let us now discuss the basic properties of the solutions of these equations, ignoring for the moment the expansion of the universe.[5] A useful reference on this subject is ref. 9.

Without a magnetic field there is only the ordinary sound wave involving density fluctuations and longitudinal velocity fluctuations (i.e., along the wave vector). In the presence of a magnetic field, however, there are no less than three different waves:

1. Fast magnetosonic waves. In the limit of small magnetic fields these waves become the ordinary sound waves. Their velocity c_+ is given by

$$c_+^2 \sim c_S^2 + v_A^2 \sin^2\theta \tag{26}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$. Fast magnetosonic waves involve fluctuations in the velocity, density, magnetic field, and gravitational field. The velocity and density fluctuations are out of phase by $\pi/2$. Equation (26) is valid for $v_A \ll c_S$. For such fields the wave is approximatively longitudinal.

2. Slow magnetosonic waves. Like the fast waves, the slow waves involve both density and velocity fluctuations. The velocity is, however, fluctuating both longitudinally and transversely even for small fields. The velocity of the slow waves is approximately

$$c_-^2 \sim v_A^2 \cos^2\theta \tag{27}$$

3. Alfvén waves. For this kind of wave $\mathbf{B}_1$ and $\mathbf{v}_1$ lie in a plane perpendicu-

[5]The full solutions are given in ref. 11.

lar to the plane through **k** and $\mathbf{B}_0$. In contrast to the magnetosonic waves, the Alfvén waves are purely rotational, thus they involve no density fluctuations. Alfvén waves are linearly polarized. Their velocity of propagation is

$$c_A^2 = v_A^2 \cos^2\theta \tag{28}$$

One should note that for v_A comparable to both c_S and the speed of light, the formula for the velocity of the Alfvén waves remains uncorrected, while the velocity of the magnetosonic waves is given by

$$c_\pm^2 = \frac{c_S^2(1 + v_A^2\cos^2\theta/c^2) + v_A^2 \pm ((c_S^2(1 + v_A^2\cos^2\theta/c^2) - v_A^2)^2 + 4v_A^2c_S^2\sin^2\theta/c^2)^{1/2})}{2(1 + v_A^2/c^2)} \tag{29}$$

8. EFFECTS ON THE CMB

The fluctuations in the CMB can be divided into primary and secondary fluctuations. The primary fluctuations involve effects coming directly from the density fluctuations and also from Doppler shifts from velocity fluctuations and gravitational redshifts.

We will concentrate on these primary effects and show that the presence of a magnetic field will change the predicted spectrum of fluctuations by changing the speed of sound.

8.1. The Fast Magnetosonic Waves

The simplest and most important case is the fast wave. Let us consider the equations describing the oscillating baryon and photon fluid in conformal Newtonian gauge using conformal time (see, e.g., ref. 12 for the case without magnetic field. They are (for small v_A)

$$\delta_b + V_b - 3\phi = 0 \tag{30}$$

$$\dot{V}_b + \frac{\dot{a}}{a} V_b - c_b^2k^2\delta_b + k^2\psi + \frac{an_e\sigma_T(V_b - V_\gamma)}{R} - \frac{1}{4\pi\hat{\rho}_b\, a}\mathbf{k}\cdot(\hat{\mathbf{B}}_0 \times (\mathbf{k} \times \hat{\mathbf{B}}_1)) = 0 \tag{31}$$

$$\delta_\gamma + \frac{4}{3} V_\gamma - 4\phi = 0 \tag{32}$$

and

$$\dot{V}_\gamma - k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) - k^2\psi - an_e\sigma_T(V_b - V_\gamma) = 0 \tag{33}$$

where $V = i\mathbf{k}\cdot\mathbf{v}$ and $R = 3\rho_b/4\rho_\gamma$. Here c_b is the baryon sound velocity in the absence of interactions with the photon gas. We have also for convenience defined $\rho_b = \hat{\rho}_b/a^3$ and $\mathbf{B} = \hat{\mathbf{B}}/a^2$. The terms with σ_T are due to Thompson scattering and couple the photons and the baryons. This term can be eliminated between the equations. If, furthermore, tight coupling is assumed (implying, e.g., $V_b \sim V_\gamma$), one can derive an equation for the density fluctuations only. If $c_b \sim 0$, one finds that in the absence of magnetic fields the effective sound velocity is

$$c_S^2 = \frac{1}{3}\frac{1}{1 + R} \tag{34}$$

Thus, through tight coupling the photons provide the baryon fluid with a pressure term and a nonzero sound velocity arises.

With a magnetic field we need one more equation:

$$\hat{\mathbf{B}}_1 = i(\hat{\mathbf{B}}_0 \cdot \mathbf{k})\mathbf{v}_b - i(\mathbf{k} \cdot \mathbf{v}_b)\hat{\mathbf{B}}_0 \tag{35}$$

Assuming longitudinal waves, we find the last term of equation (31) to be

$$-v_A^2 \sin^2\theta k^2\delta_b \tag{36}$$

as expected from the previous section.

Hence we find, to this order of approximation, that the only effect of the magnetic field is a change in the speed of sound. A simple way to account for a magnetic field is therefore to change

$$c_b^2 \to c_b^2 + v_A^2 \sin^2\theta \tag{37}$$

We have computed the microwave background spectrum with this adjustment of the sound velocity using the code of ref. 13.

An extra step in the calculation of the CMB anisotropy arises due to the fact that the velocity of the fast waves depends on the angle between the wave vector and the magnetic field. As mentioned previously, we are assuming a magnetic field that is varying in direction on scales larger than the scale of the fluctuation. Hence we should sum over all wave vectors with the angle between the magnetic field and the line of sight fixed. Different patches of the sky might therefore show different fluctuation spectra depending on this angle. In this paper we will only be considering an all-sky average assuming a field that is varying in direction on very large scales. For this reason we also sum over the angle between the field and the line of sight. In practice, it is easier to reverse the order of the sum and the calculation of the microwave background anisotropy.

We have assumed a magnetic field that gives a maximum increase in c_S^2 of $0.05c^2$ at last scattering, i.e., $v_A^2 \sim 0.05c^2$. This corresponds to 2×10^{-7} G today. For a comparison consider Fig. 2, which shows the effect of a 20% decrease of baryons. Around the first peak the effects are comparable. This allows us to obtain a rough estimate for the magnitude of the magnetic fields which should be able to be detected by future measurements of the microwave background anisotropy. The process of parameter determination using a maximum likelihood fit of the observed multipole coefficients is discussed in refs. 10 and 14. Assuming knowledge of the other cosmological parameters which affect the microwave background spectrum, a prediction of Ω_b accurate to the order of 1% or so

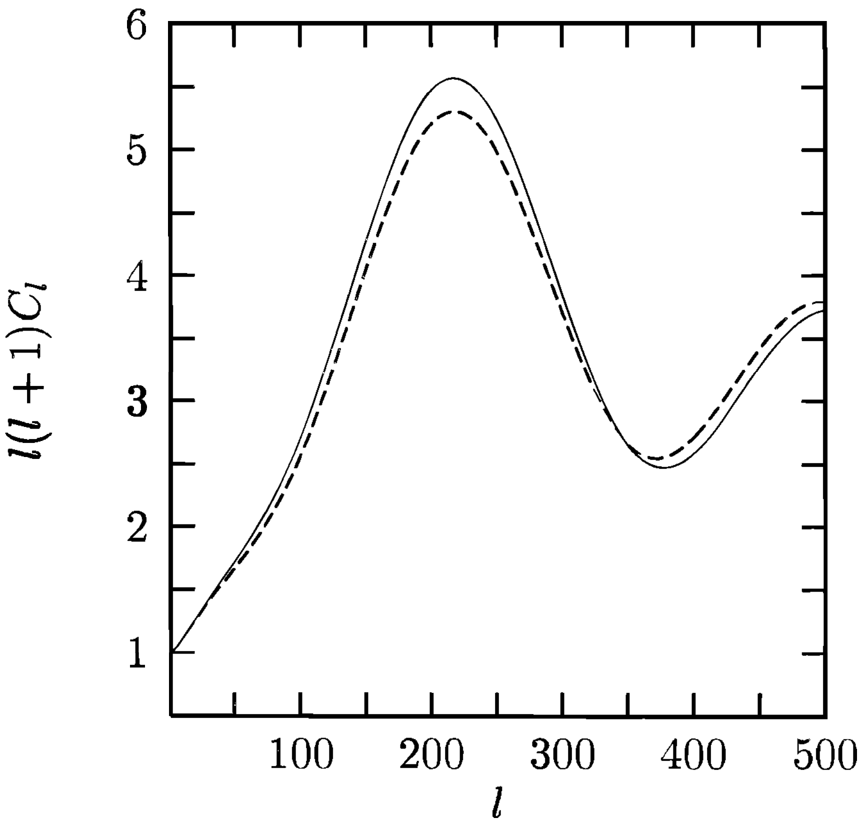

Fig. 2. The effect of lowering the baryon fraction by 20%.

should be obtainable. This translates into a limit on the current strength of magnetic fields which were present in the early universe, of the order of 5×10^{-8} G.

On very large scales, larger than the characteristic scale of the magnetic field, the effect will presumably be averaged out and the precise shape of the curve will depend on this scale. The curve in Fig. 2 is therefore not applicable for the very lowest values of l if we assume a field varying on, say, the horizon scale.

The approximations we have used can only be trusted for large scales, which means late times for the kinds of fields we are considering. For earlier times the fields are too strong and the Alfvén velocity too high. It is therefore possible that an accurate treatment of the waves might turn up even more pronounced effects at small scales.

8.2. The Slow Waves

These waves are a little bit more complicated to handle than the fast ones, even at low magnetic fields because the equations do not decouple in a simple way. The reason is that they involve both longitudinal and transverse velocity fluctuations.

It is interesting to note, however, that depending on initial conditions, they should be excited with an amplitude fixed relative to the fast waves. To illustrate this point we will consider a rather naive toy model. Using the initial conditions $\delta(0) = 0$ and $\mathbf{v} = 0$, we find (using WKB)

$$\rho \sim \alpha_+ \cos\omega_+ t + \alpha_- \cos\omega_- t + \text{const} \tag{38}$$

where $\omega_\pm = c_\pm k$. To fix the ratio α_-/α_+ we need one further initial condition on $\mathbf{B}_1$. It is reasonable to assume

$$\mathbf{B}_1(0) = 0 \tag{39}$$

i.e., all fluctuations of the magnetic field (on this scale) are due to fluctuations of the plasma initiated when entering the horizon. Using ref. 9, one can show that

$$\alpha_-/\alpha_+ \sim v_A^2/c_S^2 \tag{40}$$

Since the velocity of the slow waves is much smaller than the velocity of the fast waves for small fields, we conclude that the Doppler peaks should have a long-period modulation. Further details will be presented in a future publication.

8.3. Alfvén Waves

As discussed in the previous section, the Alfvén waves are purely rotational and involve no fluctuations in the density of the photon and baryon fluids.

With initial conditions like the ones above one sees that the Alfvén waves will not be excited. However, one could reverse the reasoning and use these waves to probe the initial conditions. They should be well suited for the detection of turbulent, rotational velocity perturbations in the early universe such as those that might be generated from primordial phase transitions. Isocurvature initial conditions are probably the most suitable to excite the Alfvén waves.

The equation describing the waves are

$$\delta_b = 0 \tag{41}$$

$$\dot{\mathbf{v}}_b + \frac{\dot{a}}{a}\mathbf{v}_b + \frac{an_e\,\sigma_T(\mathbf{v}_b - \mathbf{v}_\gamma)}{R} - i\,\frac{(\mathbf{k}\cdot\hat{\mathbf{B}}_0)}{4\pi\hat{\rho}_b a}\,\hat{\mathbf{B}}_1 = 0 \tag{42}$$

$$\delta_\gamma = 0 \tag{43}$$

and

$$\dot{\mathbf{v}}_\gamma - an_e\sigma_T(\mathbf{v}_b - \mathbf{v}_\gamma) = 0 \tag{44}$$

As expected, in this case the photon velocity is only affected by the baryon velocity through Thompson scattering.

It is evident that Alfvén waves give rise only to a Doppler effect on the CMB. As with the slow waves, we do not present any numerical estimate of the effects of the Alfvén waves. This will be done in detail in a forthcoming paper. Here we only wish to point out that since we do not have any cancellation between Doppler and gravitational effects for this kind of wave, they could provide a more clear signature of the presence of magnetic fields at the last scattering surface.

9. CONCLUSIONS ON CMBR FIELDS

These calculations have taken some preliminary steps toward understanding the effects of magnetic fields on the CMB.

We have found that the limits one can set are comparable or better than what can be achieved by other means, for example nucleosynthesis [15]. Fields below $10^{-7}/a^2$ G should be accessible in planned experiments. The possibility of finding anisotropies in different sectors of the sky and determining their nature is exciting. Depending on the scales, this may yield information on the age of these fields and their spatial extent.

We have been considering magnetic fields on scales larger than the characteristic wavelengths of the acoustic waves. It is also important to investigate the possible effects due to random fields on smaller scales.

Clearly it is important to study these possible effects in more detail and thereby take advantage of the upcoming precise measurements of the cosmic microwave background.

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